where g is the force of gravity, and σ, which is supposed to be greater than ρ, the density and a the radius of the sphere.

Hence,

$$6\pi\mu'\rho a V = \tfrac{4}{3}\pi g a^3(\sigma - \rho),$$

or

$$V = \frac{2}{9}\frac{g}{\mu'}\left(\frac{\sigma}{\rho} - 1\right)a^2 = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

by placing

$$\frac{\sigma - \rho}{\mu\rho} = \frac{\sigma}{\eta}.$$

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

CONCERNING TWO FIFTH-POWER PROBLEMS IN DIOPHANTINE ANALYSIS.

By CYRUS B. HALDEMAN, Ross, Butler County, Ohio.

I. *To resolve*

$$p^5 + q^5 + p^5 + s^5 = t^5 + u^5 + v^5 \tag{1}$$

where p, q, r, s, t, u, and v are rational.

Solution. Let $p = 2x$, $q = y - x$, $r = 2a - b^2 - x$, $s = 2a + b^2 + x$, $t = x + y$, $u = 2a - b^2 + x$ and $v = 2a + b^2 - x$.

Substituting in (1) we get

$$\begin{aligned}(2x)^5 + (y - x)^5 + (2a - b^2 - x)^5 + (2a + b^2 + x)^5 \\ = (x + y)^5 + (2a - b^2 + x)^5 + (2a + b^2 - x).^5\end{aligned} \tag{2}$$

We may write (2):

$$\begin{aligned}(x + y)^5 + (x - y)^5 + [(2a + b^2) - x]^5 - [(2a + b^2) + x]^5 \\ + [(2a - b^2) + x]^5 - [(2a - b^2) - x]^5 - (2x)^5 = 0.\end{aligned} \tag{3}$$

Expanding, regarding each binomial in parenthesis as a single quantity, and adding, we have from (3)

$$\begin{aligned}2x^5 + 20x^3y^2 + 10xy^4 - 2x^5 - 20x^3(2a + b^2)^2 - 10x(2a + b^2)^4 \\ + 2x^5 + 20x^3(2a - b^2)^2 + 10x(2a - b^2)^4 - 32x^5 = 0.\end{aligned} \tag{4}$$

Equation (4) reduces to

$$2x^2y^2 + y^4 - 16ab^2x^2 - 64a^3b^2 - 16ab^6 - 3x^4 = 0.$$

Solving for y^2, we get

$$y^2 + x^2 = \sqrt{4x^4 + 16ab^2x^2 + 64a^3b^2 + 16ab^6} = 2x^2 + 4a(b^2 + d^2), \text{ say.}$$

From this we find

$$x^2 = \frac{4a^2b^2 - (b^2 + d^2)^2a + b^6}{d^2},$$

and

$$y^2 = \frac{4a^2b^2 - (b^2 + d^2)^2a + b^6 + 4d^2(b^2 + d^2)a}{d^2}.$$

To rationalize the expressions for x and y, let

$$d^2x^2 = 4a^2b^2 - (b^2 + d^2)^2\, a + b^6 = (n - 2ab)^2, \tag{5}$$

and

$$d^2y^2 = 4a^2b^2 - (b^2 + d^2)^2\, a + b^6 + 4d^2\,(b^2 + d^2)\, a = (n - 2ab - Z)^2. \tag{6}$$

From (5) we get

$$a = \frac{n^2 - b^6}{4bn - (b^2 + d^2)^2},$$

and from (6), since the square of $n - 2ab$ is assumed equal to the first four terms of the left-hand member, we have

$$a = \frac{2nZ - Z^2}{4bZ - 4d^2\,(b^2 + d^2)}.$$

Equating the values of a and clearing of denominators, we get

$$4b^6d^2\,(b^2 + d^2) - 4b^7Z - 4d^2\,(b^2 + d^2)n^2$$
$$= Z^2\,(b^2 + d^2)^2 - 4bZ^2\, n + 4bZn^2 - 2Z\,(b^2 + d^2)^2\, n.$$

Place $4bZn^2 = -4d^2\,(b^2 + d^2)n^2$ and solve for

$$Z = -\frac{d^2\,(b^2 + d^2)}{b};$$

then

$$n = \frac{4b^7Z - 4b^6d^2\,(b^2 + d^2) + Z^2\,(b^2 + d^2)^2}{4bZ^2 + 2Z\,(b^2 + d^2)^2}.$$

Substituting the value of Z in n gives

$$n = \frac{d^2\,(b^2 + d^2)^3 - 8b^8}{2b\,(d^4 - b^4)}.$$

The value of Z and the value of n in terms of b and d, substituted in either of the above values of a gives

$$a = \frac{d^4(b^2 + d^2)^2 - 4b^8}{4b^2(d^4 - b^4)}.$$

From (5) and (6) we have

$$x = \frac{n - 2ab}{d},$$

and

$$y = \frac{n - 2ab - Z}{d}.$$

The values of a, n and Z substituted in these expressions for x, and y, give

$$x = \frac{bd^2(b^2+d^2)^2 - 4b^7}{2d\,(d^4-b^4)} \quad \text{and} \quad y = \frac{bd^2(b^2+d^2)^2 - 4b^7}{2d(d^4-b^4)} + \frac{d(b^2+d^2)}{b}.$$

From the values of a, x and y we obtain

$$p = \frac{bd^2(b^2+d^2)^2 - 4b^7}{d(d^4-b^4)}, \qquad q = \frac{d(b^2+d^2)}{b},$$

$$r = \frac{2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)}{2b^2d(d^4-b^4)},$$

$$s = \frac{2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)}{2b^2d\,(d^4-b^4)},$$

$$t = \frac{d^4(b^2+d^2) - 4b^8}{bd(d^4-b^4)},$$

$$u = \frac{2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)}{2b^2d(d^4-b^4)},$$

$$v = \frac{2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)}{2b^2d(d^4-b^4)}.$$

By substitution of the expressions just found for p, q, r, s, t, u and v in (1); and after clearing of denominators, we have the identity

$$\begin{aligned}
&[2b^3d^2(b^2+d^2)^2 - 8b^9]^5 + [2bd^2(b^2+d^2)(d^4-b^4]^5 \\
&\qquad + [2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)]^5 \\
&\qquad + [2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)]^5 \\
&= [2bd^4\,(b^2+d^2)^2 - 8b^9]^5 \\
&\qquad + [2b^4d(b^4-d^4) + d^2(b^2+d^2)^2(b^3+d^3) - 4b^8(b+d)]^5 \\
&\qquad + [2b^4d(d^4-b^4) + d^2(b^2+d^2)^2(d^3-b^3) + 4b^8(b-d)]^5
\end{aligned}$$

Take $b = 1$, $d = 2$, and we have, after dividing by 12^5,

$$16^5 + 50^5 + 53^5 + 79^5 = 63^5 + 66^5 + 69^5.$$

II. *To resolve the equality*

$$\begin{aligned} p^5 + q^5 + r^5 + s^5 + t^5 + u^5 + v^5 + w^5 \\ = (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \ldots + (2g_n)^5, \end{aligned} \tag{1}$$

where $p, q, r, s, t, u, v, w, g_1, g_2, g_3, \ldots, g_n$ *are rational.*

Solution. Let $p = a + b + c$, $q = a - b - c$, $r = b - a - c$, $s = c - a - b$, $t = d + e + f$, $u = d - e - f$, $v = e - d - f$, $w = f - d - e$.

Then, by substitution in (1), we have

$$\begin{aligned}(a + b + c)^5 &+ (a - b - c)^5 + (b - a - c)^5 + (c - a - b)^5 \\ &+ (d + e + f)^5 + (d - e - f)^5 + (e - d - f)^5 + (f - d - e)^5 \qquad (2) \\ &= (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \dots + (2g_n)^5.\end{aligned}$$

For convenience, we may write (2):

$$\begin{aligned}[a + (b + c)]^5 &+ [a - (b + c)]^5 + [(b - c) - a]^5 - [(b - c) + a]^5 \\ &+ [d + (e + f)]^5 + [d - (e + f)]^5 + [(e - f) - d]^5 - [(e - f) + d]^5 \qquad (3) \\ &= (2g_1)^5 + (2g_2)^5 + (2g_3)^5 + \dots + (2g_n)^5.\end{aligned}$$

Expanding, regarding each binomial in parenthesis as a single quantity and adding, we get from (3)

$$\begin{aligned}2a^5 &+ 20a^3(b + c)^2 + 10a(b + c)^4 - 2a^5 - 20a^3(b - c)^2 - 10a(b - c)^4 \\ &+ 2d^5 + 20d^3(e + f)^2 + 10d(e + f)^4 - 2d^5 - 20d^3(e - f)^2 - 10d(e - f)^4 \qquad (4) \\ &= 32\,(g_1^5 + g_2^5 + g_3^5 + \dots + g_n^5).\end{aligned}$$

Equation (4) reduces to

$$\begin{aligned}10a^3[(b + c)^2 - (b - c)^2] &+ 5a[(b + c)^4 - (b - c)^4] \\ &+ 10d^3[(e + f)^2 - (e - f)^2] + 5d\,[(e + f)^4 - (e - f)^4] \\ &= 16(g_1^5 + g_2^5 + g_3^5 + \dots + g_n^5);\end{aligned}$$

and by further expansion and reduction, we have

$$5(a^3bc + ab^3c + abc^3 + d^3ef + de^3f + def^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5). \qquad (5)$$

Let $d = a$ and (5) becomes

$$5a^3\,(bc + k^3\,ef) + 5a\,(b^3c + bc^3 + ke^3f + kef^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5).$$

Place $bc + k^3ef = 0$ and have

$$5a(b^3c + bc^3 + ke^3f + kef^3) = 2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5).$$

From these equations we obtain

$$b = -\frac{k^3ef}{c}, \qquad a = \frac{2(g_1^5 + g_2^5 + g_3^5 + \cdots + g_n^5)}{5(b^3c + bc^3 + ke^3f + kef^3)},$$

which give the requirements.